Notes in preparation to the summer school

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Contents

Introduction and other stuff

These notes are an expanded account of two informal talks I gave locally in Göttingen, in preparation for the summer school ”Higher Categories for the Working Mathematician”, of which I am an organizer. I decided to type these as it was both helpful in preparing the talks, and in the hope that they will be useful to at least someone looking to have a quick read in preparation to the main lectures of the school, or to remind themselves what are the very basics we’re starting from.

0.1 Contents

The first talk discusses simplicial sets, omitting lots of technical details due to the limited time that was available for the talk, and instead focusing a lot on examples. The main ones are the Nerve of a category and the Singular simplicial set of a topological space. These notions are foundational in the approach to ∞-categories via quasicategories, due to Joyal and Lurie.

The second talk discusses the two main classical motivations that lead to the development of spectra, and then does a quick overview of the various classical models available, the properties of the stable homotopy category, pinpointing here and there the advantages of using ∞-categories to treat spectra and to do stable homotopy theory.

0.2 Notes for the reader

As of July 27, 2026, the notes are still incomplete in the last section as I haven’t had time to type that yet, and there’s some pictures missing that I hope to add soon. Moreover, I still haven’t gone through them thoroughly a second time, so there’s probably plenty of typos, though hopefully no conceptual mistakes. They are being uploaded now as it’s hopefully not too late for interested participants to take a look!

0.3 Claims

There is no claim of originality in the content of these notes. The presentation of the material is original, though probably not that much different with any other introductory material or survey on these topics. Some useful materials that were used when preparing the talks are [Lan21] for the talk on simplicial sets, and [Mal14] for the talk on spectra.

No AI was used for the typesetting of this document, the writing or the nice pictures. I only used it occasionally to remind myself of some latex commands (using google to search for specific tikz commands does get admittedly quite annoying!) and to sometimes do some soundness checks.

1 Simplicial sets, Nerve and Sing

In this first talk we give the definition of a simplicial set and we look at tons of examples. Most importantly, we look at the Nerve simplicial set of a category and at the Singular simplicial set of a topological space.

1.1 Simplicial sets

Let us start with recalling some basic definitions from category theory. We shall not recall everything (as it is usually customary in such short talks that do assume some background), and similarly we shall not worry at all about size issues; we can for example assume the large cardinals axiom.

Definition 1.1. A *category* $\mathcal{C}$ is given by the following data:

- a (possibly large) set of *objects* $\mathrm{ob}(\mathcal{C})$;
- for any two objects x, y of $\mathcal{C}$, a (possibly large) set of *morphisms* $\mathrm{Hom}_{\mathcal{C}}(x, y)$;

equipped with the following:

- for any object x in $\mathcal{C}$, a chosen morphism $\mathrm{id}_x \in \mathrm{Hom}_{\mathcal{C}}(x, x)$, called the *identity* of x;
- for any three objects x, y, z in $\mathcal{C}$, a composition law (map of sets)

$$\mathrm{Hom}_{\mathcal{C}}(x, y) \times \mathrm{Hom}_{\mathcal{C}}(y, z) \to \mathrm{Hom}_{\mathcal{C}}(x, z).$$

Composition and identities are required to satisfy associativity and unitality.

The natural notion of a ”map” between categories is that of a functor:

Definition 1.2. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ from $\mathcal{C}$ to $\mathcal{D}$ is given by:

- a map of sets $\mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D})$;
- for any two objects x, y of $\mathcal{C}$, a map of sets $\mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(F(x), F(y))$.

(we will generally abuse notation and denote both of the above also by F). The latter maps are required to be compatible with compositions and units. That is, we should have $F(g \circ f) = F(g) \circ F(f)$ and $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$.

Let us look at some examples:

Example 1.3. Your favourite mathematical objects, together with an appropriate notion of maps between them, can most certainly be collected into a category. For example, there are categories of sets, monoids, (abelian) groups, (any kinds of) rings, modules, but also topological spaces, smooth manifolds, banach spaces, ...

Moreover, your favourite constructions that go from one category to another will most certainly give rise to functors. For example, the homology groups of a topological space X with coefficients in $\mathbb{Z}$ can be seen as functors H^n from the category of topological spaces to the one of abelian groups.

Example 1.4. Given a category $\mathcal{C}$, we can construct another category $\mathcal{C}^{\mathrm{op}}$, called the *opposite category* of $\mathcal{C}$, by taking the same objects as $\mathcal{C}$ and setting $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(x, y) := \mathrm{Hom}_{\mathcal{C}}(y, x)$ for any two objects x, y. One can check that compositions and identities are well-defined and still associative and unital. Morally, we are ”changing the direction” of all arrows in a compatible way.

Example 1.5. Given two categories $\mathcal{C}$ and $\mathcal{D}$, there is a category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ with objects being functors $\mathcal{C} \to \mathcal{D}$, and morphisms being natural transformations[1] between them.

Here’s one more example:

Example 1.6. A poset (Partially Ordered Set) is a set P together with a *order* relation $\leq$ that is reflexive, antisymmetric, and transitive. A map of posets $P \to Q$ is a map of sets that preserves the order relation. posets and maps of posets can be collected into a category, denoted by PoSet.

Now, a full subcategory of PoSet that will be very important for us is the following:

[1]We didn’t define these, but ”trust me it can be done” (i.e. check any book on basic cat theory)

Definition 1.7. The category Δ is defined as follows:

- for any $n \in \mathbb{N}$ (0 included!) there is an object $[n]$, which should be thought of as the poset:

$$[n] = \{0 \leq 1 \leq \cdots \leq n\};$$

- given $[n]$ and $[m]$ and viewing them as posets, the morphisms $[n] \to [m]$ are the maps of posets, i.e. the order preserving maps.

Composition laws and units are the expected ones coming from the underlying sets.

Let us look at the morphisms of Δ a bit more in detail. Given $[n-1]$ and $[n]$ in Δ ($n \geq 1$), there's a priori tons of possible morphisms between them in Δ. There are however two "special" kinds of maps, in the sense that they are quite simple to write down:

- for every $0 \leq i \leq n$, denote by $\delta_i : [n-1] \to [n]$ the unique injective order preserving map that "skips" i in the codomain $[n]$. That is, such that $\mathrm{im}(\delta_i) = [n] \setminus \{i\}$. Concretely, δ_i maps

$$0 \mapsto 0, \quad 1 \mapsto 1, \quad \dots, \quad i-1 \mapsto i-1, \quad i \mapsto i+1, \quad \dots, \quad n-1 \mapsto n.$$

- for every $0 \leq i \leq n-1$, denote by $\sigma_i : [n] \to [n-1]$ the unique surjective order preserving map that "doubles" i in the codomain $[n-1]$. That is, such that $|\sigma_i^{-1}(\{i\})| = 2$. Concretely, σ_i maps

$$0 \mapsto 0, \quad 1 \mapsto 1, \quad \dots, \quad i \mapsto i, \quad i+1 \mapsto i, \quad \dots, \quad n \mapsto n-1.$$

Remark 1.8. It turns out that these morphisms satisfy the so-called *simplicial identities*[2], and moreover any morphism in Δ can be written as a composition of σ_i's, followed by a composition of δ_j's. For example. the morphism $[2] \to [2]$ mapping $0, 1 \mapsto 0$ and $2 \mapsto 1$ is the composition of $\sigma_0 : [2] \to [1]$ with $\delta_2 : [1] \to [2]$.

So, the category Δ looks like this:

$$\begin{tikzcd}
{[0]} \arrow[r, bend left, "\delta_0"] \arrow[r, bend right, "\delta_1"'] & {[1]} \arrow[l, "\sigma_0"'] \arrow[r, bend left=50, "\delta_0"] \arrow[r, "\delta_1"] \arrow[r, bend right=50, "\delta_2"'] & {[2]} \arrow[l, bend right=25, "\sigma_0"'] \arrow[l, bend left=25, "\sigma_1"] & \vdots & {[3]} & \cdots
\end{tikzcd}$$

Now, we are in the position to define simplicial sets.

Definition 1.9. A *simplicial set* is a functor $\Delta^{\mathrm{op}} \to \mathrm{Set}$. The category sSet of simplicial sets is the functor category $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set})$.

Concretely, a simplicial set X is given by the following data:

- for every $n \in \mathbb{N}$, a set $X_n := X([n])$. Its elements are the *n-simplices* of X;
- for every $n \geq 1$ and $0 \leq i \leq n$, maps $d_i := X(\delta_i) : X_n \to X_{n-1}$, called *face maps*;
- for every $n \geq 1$ and $0 \leq i \leq n-1$, maps $s_i := X(\sigma_i) : X_{n-1} \to X_n$, called *degeneracy maps*;

where the maps d_i and s_j satisfy the cosimplicial identities (dual to the simplicial identities in Δ). X looks like this:

$$\begin{tikzcd}
X_0 \arrow[r, "s_0"] & X_1 \arrow[l, bend right, "d_0"'] \arrow[l, bend left, "d_1"] \arrow[r, bend left=25, "s_0"] \arrow[r, bend right=25, "s_1"'] & X_2 \arrow[l, bend right=50, "d_0"'] \arrow[l, "d_1"'] \arrow[l, bend left=50, "d_2"] & \vdots & X_3 & \cdots
\end{tikzcd}$$

Notice how the arrows go the opposite way compared to the ones in Δ, as simplicial sets are functors from Δ^{op}. We also define the following; the motivation for it will become clear when looking at examples, as looking at nondegenerate simplices gives a nice graphical intuition.

Moreover, the category sSet is a functor category, which in particular means that a morphism of simplicial sets is a natural transformation of functors $\Delta^{\mathrm{op}} \to \mathrm{Set}$. Writing it down explicitly in this case, it means that a morphism of simplicial sets $f : X \to Y$ is a collection of maps $f_n : X_n \to Y_n$ that are compatible with all face and degeneracy maps.

[2] I won't write them here for now, a quick google search will find them

Definition 1.10. If $x \in X_n$ is in the image of some s_i for any n, i, then x is called *degenerate*.

Before looking at examples, recall that the Yoneda lemma applied to Δ gives a fully fauithful functor $\Delta \hookrightarrow \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set}) =: \mathrm{sSet}$, called the *Yoneda embedding*. At the level of objects, this maps $[n]$ to the presheaf $\mathrm{Hom}_\Delta(-, [n])$. Such presheaves, in the image of the Yoneda embedding, are called *representable presheaves*.

Let us now look at lots of examples of simplicial sets.

Example 1.11. For every $n \in \mathbb{N}$, let $\Delta^n := \mathrm{Hom}_\Delta(-, [n]) \in \mathrm{sSet}$. We will refer to Δ^n as the *standard n-simplex* in sSet. Let's look at how Δ^n looks like for small values of n.

$n = 0$: By definition, $\Delta^0 := \mathrm{Hom}_\Delta(-, [0])$, so $(\Delta^0)_n = \mathrm{Hom}_\Delta([n], [0])$. For every n, there is always exactly one order preserving map to $[0] = \{0\}$, therefore $(\Delta^0)_n = \{*\}$ is a singleton for every n. All of the maps d_i and s_j must be id_*. So, Δ^0 looks like this:

$$\begin{tikzcd}[column sep=large] \{*\} \arrow[r, "\mathrm{id}_*" description] & \{*\} \arrow[l, bend right, "\mathrm{id}_*"'] \arrow[l, bend left, "\mathrm{id}_*"] \arrow[r, bend left=15, "\mathrm{id}_*"] \arrow[r, bend right=15, "\mathrm{id}_*"'] & \{*\} \arrow[l, bend right=40, "\mathrm{id}_*"'] \arrow[l, "\mathrm{id}_*" description] \arrow[l, bend left=40, "\mathrm{id}_*"] & \vdots & \{*\} & \cdots \end{tikzcd}$$

Note that the only nondegenerate simplex is the $*$ in $(\Delta^0)_0$, as all others are in the image of all s_i's. Graphically[3], we may therefore think of Δ^0 as a single point:

Δ^0 :

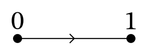

$n = 1$: By definition, $\Delta^1 := \mathrm{Hom}_\Delta(-, [1])$, so $(\Delta^1)_n = \mathrm{Hom}_\Delta([n], [1])$. Now there's several order preserving maps from $[n]$ to $[1]$; one way to write them down easily is as follows. Writing down the images in order gives a sequence $0 \dots 01 \dots 1$ with $n + 1$ total digits: this will always be some number of zeroes. followed by all ones. So, for any n, $\mathrm{Hom}_\Delta([n], [1])$ has exactly $n + 2$ elements:

$$\mathrm{Hom}_\Delta([n], [1]) = \{0 \dots 0, 0 \dots 01, 0 \dots 011, \dots, 01 \dots 1, 1 \dots 1\}.$$

The maps d_i delete the digit in the i-th position, and the maps s_j double (i.e. write down twice) the digit in the j-th position. So, Δ^1 looks like this:

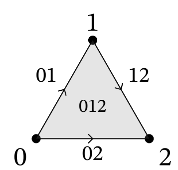
$$\begin{tikzcd}[column sep=large] \{0,1\} \arrow[r, "s_0" description] & \{00, 01, 11\} \arrow[l, bend right, "d_0"'] \arrow[l, bend left, "d_1"] \arrow[r, bend left=15, "s_0"] \arrow[r, bend right=15, "s_1"'] & \{000, 001, 011, 111\} \arrow[l, bend right=40, "d_0"'] \arrow[l, "d_1" description] \arrow[l, bend left=40, "d_2"] & \cdots \end{tikzcd}$$

For example, the map s_0 sends $0 \mapsto 00$ and $1 \mapsto 11$, the map d_0 sends $00 \mapsto 0$, $01 \mapsto 1$ and $11 \mapsto 1$, whereas the map d_1 sends $00 \mapsto 0$, $01 \mapsto 0$ and $11 \mapsto 1$. It is easy to check that the nondegenerate simplices are $0, 1$ and 01. Graphically, Δ^1 may be depicted as:

Δ^1 :


where, in order to draw it, d_0 and d_1 should be thought of as "target" and "source" maps respectively. They tell us that the source of the 1-simplex 01 should be $0 = d_1(01)$, and the target should be $1 = d_0(01)$.

$n = 2$: This is totally analogous, just more complicated. For example, n-simplices of Δ^2 can be identified with strings of digits $0 \dots 01 \dots 12 \dots 2$ of length $n + 1$. The nondegenerate simplices are $0, 1, 2, 01, 02, 12$ and 012, leading to the following graphical depiction:

Δ^2 :


Note for example that d_0, d_1, d_2 applied to the 2-simplex 012 give its three faces. d_i gives the face that is opposite to the vertex i.

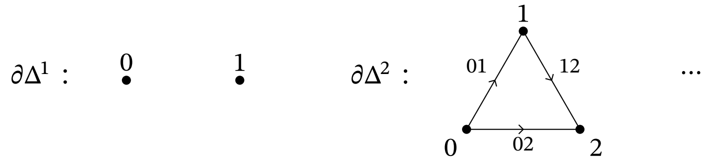
[3]additional motivation for why these pictures are a correct thing to look at will become clear once we discuss the geometrical realization

Example 1.12. Here are some more examples of important simplicial sets[4].

- for every $n \in \mathbb{N}$, we can define the *boundary* simplicial sets $\partial\Delta^n \subseteq \Delta^n$. Its m-simplices are given as follows:

$$(\partial\Delta^n)_m = \{\text{non-surjective order preserving maps } [m] \to [n]\} \subseteq \mathrm{Hom}_\Delta([m],[n]) = (\Delta^n)_m.$$

  Note that $\partial\Delta^0 = \emptyset$. Graphically, the boundaries for low n look like this:

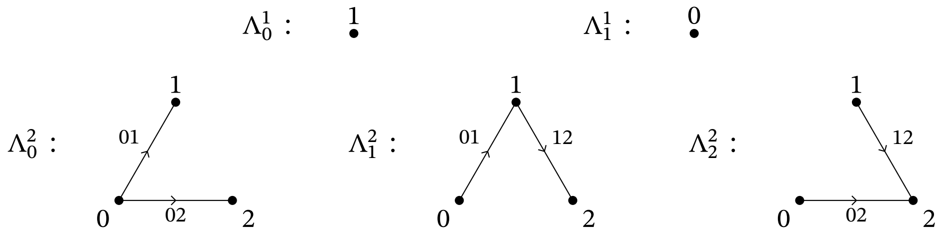


  Intuitively, we are removing the unique nondegenerate n-simplex from Δ^n (and various other degenerate simplices).

- for every $n \geq 1$ and every $0 \leq k \leq n$, define the *horn* simplicial sets $\Lambda^n_k \subseteq \Delta^n$ as follows:

$$(\Lambda^n_k)_m = \{[m] \xrightarrow{f} [n] \text{ such that there exists } i \in [n] \setminus k \text{ with } i \notin \mathrm{im}(f)\} \subseteq \mathrm{Hom}_\Delta([m],[n]) = (\Delta^n)_m.$$

  Graphically, the horns for $n = 1, 2$ look like this:



  Here, again intuitively, from Δ^n we are removing the unique nondegenerate n-simplex, together with the $n-1$-simplex corresponding to the face opposite to the vertex i.

- for every $n \in \mathbb{N}$, we define the *spine* simplicial sets $I^n \subseteq \Delta^n$ as follows:

$$(I^n)_m = \{[m] \xrightarrow{f} [n] \text{ such that } \mathrm{im}(f) \text{ is either } \{j\} \text{ or } \{j, j+1\} \text{ for some } j\} \subseteq \mathrm{Hom}_\Delta([m],[n]) = (\Delta^n)_m.$$

  One has that $I^0 = \Delta^0$, $I^1 = \Delta^1$, $I_2 = \Lambda^2_1$, and in general, for large n, I^n is given by all vertices, linked by the edges $01, 12, 23, \ldots$.

And finally, another useful thing to know is the following example/definition:

Definition 1.13. Given $X \in \mathrm{sSet}$, its *k-skeleton* $\mathrm{sk}_k X$ is defined to be the smallest subsimplicial set of X containing all nondegenerate n-simplices of X for $n \leq k$.

Some words in the above definition were not defined, but you can already think of the skeleton informally. For example, the 1-skeleton of Δ^2 is $\partial\Delta^2$.

Finally, let us list some key facts for simplicial sets, without proof. This is intentionally very dry as the focus of the talk is more on examples, the best I can offer is my apologies. I might elaborate on these in a future version of these notes.

Facts 1.14. *Let* $X \in \mathrm{sSet}$.

1. *By the Yoneda lemma,* $X_n \cong \mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, X)$. *This means that we can view n-simplices of* X *as maps of simplicial sets* $\Delta^n \to X$.
2. $X \cong \mathrm{colim}_{[n]\in\Delta/X}\, \Delta^n$. *This means that every simplicial set can be written as some colimit of standard n-simplexes. For example, the* 2*-horns are pushouts of two copies of* Δ^1*, glued in different fashions.*
3. sSet *is bicomplete, since* Set *is, and (co)limits are computed "pointwise" in* sSet*. This is always true for presheaf categories with values in* Set.
4. sSet *is cartesian closed. This means that we can define an internal Hom, therefore giving the sets* $\mathrm{Hom}_{\mathrm{sSet}}(X, Y)$ *a simplicial set structure themselves.*

[4] Note that we are being quite imprecise when defining them, as one should check that these really give simplicial sets by checking that the face and degeneracy maps are well-defined

1.2 The Nerve and Sing constructions

Now, the plan is to look at some more examples of simplicial sets arising from:

1. category theory;
2. topology.

And now for a little bit of foreshadowing. Simplicial sets give rise to a good model for $(\infty, 1)$-categories (∞-categories for short): the theory of quasicategories. One of the reasons this approach has been so successful is due to the simplicity with which it encompasses both ordinary category theory and topology, in a nice and compatible way. After all, the idea of ∞-categories is to have a nice categorical framework to do category theory "up to homotopy", as lots of constructions in the usual category of topological spaces do not work up to homotopy, and the homotopy category itself is ill-defined (it does not, for example, have many limits or colimits). But discussing this is not within the scope of this talk, as we hope that the main lecture series will mention this problem, at least in some level of detail.

Therefore, let's just look at how to go back and forth from categories and from topological spaces to simplicial sets.

1.2.1 The Nerve of a category

Let $\mathcal{C}$ be a category.

Construction 1.15. Construct a simplicial set $N(\mathcal{C})$ as follows. We define the n-simplices to be

$$\begin{aligned} N(\mathcal{C})_0 &= \mathrm{ob}(\mathcal{C}) = \{\bullet\}; \\ N(\mathcal{C})_1 &= \mathrm{mor}(\mathcal{C}) = \{\bullet \to \bullet\}; \\ N(\mathcal{C})_n &= \{n\text{-tuples of composable morphisms in } \mathcal{C}\} = \{\bullet \to \bullet \to \cdots \to \bullet\}. \end{aligned}$$

and we define the face and degeneracy maps to be the following

$$\begin{aligned} d_i : N(\mathcal{C})_n \to N(\mathcal{C})_{n-1} \qquad & (\cdots \bullet \xrightarrow{f} \bullet_i \xrightarrow{g} \bullet \cdots) \mapsto (\cdots \bullet \xrightarrow{g \circ f} \bullet \cdots) \quad 0 < i < n \\ & (\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \cdots) \mapsto (\bullet \xrightarrow{g} \bullet \cdots) \quad i = 0 \\ & (\cdots \bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet) \mapsto (\cdots \bullet \xrightarrow{f} \bullet) \quad i = n \\ s_i : N(\mathcal{C})_{n-1} \to N(\mathcal{C})_n \qquad & (\cdots \bullet_i \xrightarrow{f} \bullet \cdots) \mapsto (\cdots \bullet \xrightarrow{\mathrm{id}} \bullet \xrightarrow{f} \bullet \cdots) \end{aligned}$$

The face maps d_i are either composing two composable morphisms if $0 < i < n$, or they are removing the initial/terminal one when $i = 0$ or n. The degeneracy maps simply add an identity morphism in the right spot.

This gives a simplicial set (check!), called the *nerve* of $\mathcal{C}$.

Although the above works, it's quite a bit explicit and, at least on a first inspection, not super nice. Turns out, that it pops up from a more abstract procedure, which works also more generally:

Remark 1.16. Note that there is a functor $F : \Delta \to \mathrm{Cat}$ that maps $[n]$ to the category, also denoted by $[n]$, coming from the totally ordered set $[n] = \{0 \le 1 \le \cdots \le n\}$ itself. Such an F is called a *cosimplicial object* in Cat.

Then, for every n we have

$$N(\mathcal{C})_n = \mathrm{Hom}_{\mathrm{Cat}}(F([n]), \mathcal{C}) = \mathrm{Hom}_{\mathrm{Cat}}([n], \mathcal{C}) = \mathrm{Fun}([n], \mathrm{Cat})$$

and for every $f : [n] \to [m]$, the induced map

$$\mathrm{Hom}_{\mathrm{Cat}}([m], \mathcal{C}) = N(\mathcal{C})_m \to N(\mathcal{C})_n = \mathrm{Hom}_{\mathrm{Cat}}([n], \mathcal{C})$$

is given by precomposing with $F(f)$.

Exercise 1.17. Check that the functor F above can actually be defined. In other words, describe what the image under F of any $f : [n] \to [m]$ in Δ should be, and check that F so defined is a functor. This essentially amounts to checking that what we had written down explicitly earlier defines a simplicial set, but the latter is arguably a much nicer way to see it.

And here is a condensed list of important properties of the nerve.

Facts 1.18. *(You may consider any of these as Exercises, as there's no proofs here.)*

1. *$N([n]) \cong \Delta^n$ as simplicial sets.*

2. *The nerve construction is functorial, and the resulting functor $N : \mathrm{Cat} \hookrightarrow \mathrm{sSet}$ is fully faithful.*

3. *For $X \in \mathrm{sSet}$, the following are equivalent:*

   (a) $X \cong N(\mathcal{C})$ for some $\mathcal{C}$;

   (b) for every n, for every $0 < i < n$:

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

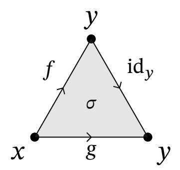
   (c) for every $n \geq 2$:

$$\begin{tikzcd} I^n \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

4. *The functor $N : \mathrm{Cat} \to \mathrm{sSet}$ has a left adjoint $\tau : \mathrm{sSet} \to \mathrm{Cat}$.*

Remark 1.19. The left adjoint τ can be obtained by left Kan extending $F : \Delta \to \mathrm{Cat}$ along the Yoneda embedding $\Delta \hookrightarrow \mathrm{sSet}$. If X is "nice enough" (i.e., if X is a quasicategory, but we haven't defined this yet), then the category $\tau(X)$ may be described as follows:

- $\mathrm{ob}(\tau(X)) = X_0$;
- $\mathrm{mor}(\tau(X)) = X_1/\sim$, where $f \sim g$ iff there is a 2-simplex $\sigma \in X_2$ with faces $d_0 = \mathrm{id}, d_1 = g, d_2 = f$. (This gives an equivalence relation precisely when X is a quasicategory) The idea is that σ is describing a "homotopy" from f to g. Such a σ would look like this:



In this case, one usually writes $h(X) := \tau(X)$ and calls it the *homotopy category* of X.

1.2.2 The Sing of a topological space

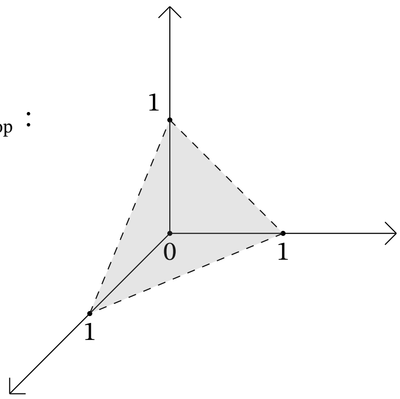
Let now T be a (nice enough) topological space. Say that Top is a category of nice enough topological spaces and continuous maps between them.

Construction 1.20. Construct a simplicial set $\mathrm{Sing}(X)$ as follows:

- $\mathrm{Sing}(X)_n := \mathrm{Hom}_{\mathrm{Top}}(\Delta^n_{\mathrm{Top}}, T)$, where $\Delta^n_{Top} \hookrightarrow \mathbb{R}^{n+1}$ with the subspace topology is the n-th topological simplex, defined as follows:

$$\Delta^n_{Top} := \left\{ (x_k)_k \in \mathbb{R}^{n+1} \text{ s.t. } \sum_k x_k = 1, x_k \geq 0 \forall k \right\};$$

Δ^2_{Top} :



- the face maps d_i are determined by the maps $\Delta^{n-1}_{\mathrm{Top}} \to \Delta^n_{\mathrm{Top}}$ which add a 0 in position i;
- the degeneracy maps s_i are determined by the maps $\Delta^n_{\mathrm{Top}} \to \Delta^{n-1}_{\mathrm{Top}}$, $(x_k)_k \mapsto (\dots, x_i + x_{i+1}, \dots)$, summing in position i.

As before, the above construction seems a bit convoluted, but it's perhaps now clearer that it comes from a functor $\Delta \to \mathrm{Top}$ mapping $[n] \mapsto \Delta^n_{\mathrm{Top}}$, in similar fashion as before.

Exercise 1.21. Check that the above construction actually gives a simplicial set. Alternatively, check that we actually have a functor $\Delta \to \mathrm{Top}$; the images of the morphisms $[n] \to [m]$ are precisely those defined in the construction, and you have to check functoriality.

And here are some facts.

Facts 1.22. *(You may again consider any of these as Exercises, as there's no proofs here.)*

1. *The* Sing *construction can be made functorial, giving a functor* $\mathrm{Sing} : \mathrm{Top} \to \mathrm{sSet}$.
2. $\mathrm{Sing}(T)$ *satisfies the following: for every* $n \geq 0$*, for every* $0 \leq i \leq n$

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & \mathrm{Sing}(T) \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

3. Sing *has a left adjoint* $|-| : \mathrm{sSet} \to \mathrm{Top}$*, called the* geometric realization. *This may also be obtained by left Kan extension, and there is an explicit formula for it: for* $X \in \mathrm{sSet}$*, we have*

$$|X| = \operatorname{colim}_{\Delta^n \in \Delta/X} \Delta^n_{\mathrm{Top}} = \left(\coprod_n X_n \times \Delta^n_{\mathrm{Top}} / \sim \right)$$

where $(f^* x, t) \sim (x, f_* t)$. *This is describing a "gluing" procedure of topological n-simplices along their boundaries.*

4. $|\Delta^n| \simeq \Delta^n_{\mathrm{Top}}$, *and in general* $|X|$ *is a CW complex. Moreover, the unit* $T \to |\mathrm{Sing}(T)|$ *of the adjunction is a weak homotopy equivalence.*

Remark 1.23. The converse to 2. is almost true, in a precise sense (it is true not up to iso, but up to homotopy equivalence in sSet). In general, Sing only remembers info up to weak homotopy equivalence.

This marks the end of the talk. As a final remark, note that the extension conditions for the Nerve and Sing simplicial sets in diagrams

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & \mathrm{Sing}(T) \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

are almost the same, except for:

- for the Nerve, we only have $0 < i < n$, so we don't have $i = 0$ or $i = n$, but we also have unicity of the extensions;
- for the Sing, we also have extensions for $i = 0, n$ but they are not necessarily unique anymore.

One has the following:

- the union of the two conditions, so $0 \leq i \leq n$ and unique extensions, leads to usual groupoids, i.e. categories where all morphisms are invertible;
- the intersection of the two conditions, so $0 < i < n$ and existence of extensions but no unicity, leads to the notion of a *quasicategory*. This turns out to be a good model for the theory of $(\infty, 1)$-categories.

2 Spectra: classical motivations and point of view

The goal of this talk is to have an introduction to spectra in the classical sense, in particular highlighting the main classical motivations that lead to the development of such ideas. This is to provide introduction and motivation to Rune Haugseng's lecture series, which will deal with spectra as the prime tool to do stable homotopy theory in an ∞-categorical context.

The plan is to look at the following:

- some classical homotopy theory, and motivation 1 for spectra coming from wanting to study spaces "up to stabilization";
- some algebraic topology, and motivation 2 for spectra coming from wanting to corepresent generalized cohomology theories;
- classical perspectives on defining a category of spectra, and main properties of the stable homotopy category.

Notation 2.1. In what follows,

- Top will denote the category of compactly generated, weak Hausdorff, topological spaces and continuous maps between them;
- CW will denote the category of CW complexes and continuous maps between them:
- Top_* and CW_* will denote their pointed variants.

Whenever I will write "topological space", I will mean an object in Top, so a compactly generated, weak Hausdorff, topological space.

2.1 Some classical homotopy theory

In this first section, we discuss some definitions and results from classical homotopy theory, as well as motivations from this angle that motivate the introduction of a "category of spectra".

Let us first look at some classical definitions.

Definition 2.2. Let X, Y be topological space, $f, g : X \to Y$ continuous maps. Then, we say that f is *homotopic* to g, written $f \sim g$, if there exists a continuous map $H : X \times I \to Y$ such that $H|_{X\times\{0\}} = f$, $H|_{X\times\{1\}} = g$.

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One can check that this gives an equivalence relation on each set of morphisms $\mathrm{Hom}_{\mathrm{Top}}(X, Y)$.

Definition 2.3. For X, Y topological spaces, define $[X, Y] := \mathrm{Hom}_{\mathrm{Top}}(X, Y)/\sim$.

Definition 2.4. Let X, Y topological spaces. We say that X is *homotopy equivalent* to Y, written $X \simeq Y$, if there exist continuous maps $f : X \to Y, g : Y \to X$ such that $g \circ f \sim \mathrm{id}_x, f \circ g \sim \mathrm{id}_y$.

Remark 2.5. All of the above definitions can be restated for X, Y *pointed* topological spaces. We obtain

- a notion of *pointed homotopy* between pointed continuous maps;
- an equivalence relation on each $\mathrm{Hom}_{\mathrm{Top}_*}(X, Y)$,. We will denote $[X, Y]_* := \mathrm{Hom}_{\mathrm{Top}_*}(X, Y)$;
- a notion of two spaces being *pointed homotopy equivalent*.

If the spaces are nice enough (path connectivity hypotheses are needed for example), then these notions coincide with the unpointed versions, but in general they do not.

Remark 2.6. One may also state all of the above definitions working in CW or in CW_*. While these two categories are different from Top and Top_*, we'll see later that the respective homotopy categories are equivalent.

Now, the sets $[X, Y]_*$ are in general very interesting, but very hard to compute. A particularly nice thing to do is choosing $X = S^n$ to be the n-th sphere, with a canonical choice of a basepoint (e.g. usually the north pole).

Definition 2.7. Let X be a pointed topological space. Then, define

$$\pi_n(X) := [S^n, X]_*.$$

We have

- $\pi_0(X) = [S^0, X]_* \cong \{\text{connected components of } X\}$ is the set of connected components of X. In general, this is just a set.
- $\pi_1(X) = [S^1, X]_*$ is the so-called *fundamental group* of X, as it turns out that it always admits a canonical group structure (but in general not abelian!).
- for $n \geq 2$, $\pi_n(X)$ always admits the structure of an abelian group. These are the so-called *homotopy groups* of X.

(It would be nice if also π_0, π_1 were abelian groups, no?)

The golden apple of classical homotopy theory is the computation of the homotopy groups of spheres, $\pi_n(S^k)$ for all possible values of $n, k \geq 0$. Only $n \geq k$ is actually interesting as for $n < k$ one always has $\pi_n(S^k) \simeq 0$ (rather easy to prove), but this is a VERY HARD problem in general, and to this day there are only very partial computations.

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So, how could one go trying to simplify the study of homotopy groups? We will now go towards one direction, which stems from the Freudenthal suspension theorem (cfr later). Let us look at a bit more classical definitions and results then to work our way towards this result.

Definition 2.8. Let X, Y be topological spaces, $f : X \to Y$ be a continuous map. Then, we say f is a *weak homotopy equivalence* if

1. f induces a bijection of sets on path-connected components $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$:
2. for any $n \geq 1$, f induces an isomorphism of groups $\pi_n(f) : \pi_n(X, x) \to \pi_n(Y, f(x))$ for any choice of basepoint $x \in X$.

Notice that being a homotopy equivalence implies being a weak homotopy equivalence. The converse does not hold for general topological spaces, but it holds for CW complexes:

Theorem 2.9 (Whitehead)**.** *Let X, Y be CW complexes, $f : X \to Y$. Then, f is a homotopy equivalence if and only if f is a weak homotopy equivalence.*

Now, one can localize (see Kalin's talk 2) the categories Top and CW, at weak homotopy equivalences and at homotopy equivalences respectively: denote by

- $\mathrm{HoTop} := \mathrm{Top}[\mathrm{w.h.eq.}^{-1}]$ the localization of Top at weak homotopy equivalences;
- $\mathrm{HoCW} := \mathrm{CW}[\mathrm{h.eq}^{-1}]$ the localization of CW at homotopy equivalences;
- HoTop_* and HoCW_* their pointed variants.[5]

So for example, the hom in HoCW will be given by $[X, Y]$ and the hom in HoCW_* will be given by $[X, Y]_*$.

Fact 2.10. *One has equivalences of categories* $\mathrm{HoTop} \simeq \mathrm{HoCW}$ *and* $\mathrm{HoTop}_* \simeq \mathrm{HoCW}_*$.

These follow essentially by CW approximation and Whitehead theorem arguments. We obtain functors

$$\begin{array}{ll} \mathrm{Top} \to \mathrm{HoTop} & \mathrm{CW} \to \mathrm{HoCW} \\ \mathrm{Top}_* \to \mathrm{HoTop}_* & \mathrm{CW}_* \to \mathrm{HoCW}_* \end{array}$$

Let's look at some explicit constructions now in the setting of pointed topological spaces.

[5]When going to the pointed setting, the latter condition for a map f to be a weak homotopy equivalence is really needed if X, Y are not path connected, and cannot be dropped.

Definition 2.11. Let X be a pointed topological space. Its *(reduced) suspension* ΣX is defined as the pointed topological space
$$\Sigma X := X \times I / X \times \{0\} \cup X \times \{1\} \cup \{*_X\} \times I$$
where the canonical choice of basepoint is given by the quotiented part.

ADDPICTURE

Intuitively, one can think of the suspension of X as the gluing of two cones over X along their boundaries. (note: cones are contractible!)

Example 2.12. $\Sigma S^0 \simeq S^1$, and in general $\Sigma S^n \simeq S^{n+1}$.

Definition 2.13. Let X be a pointed topological space. Its loop space ΩX is defined as

$$\Omega X := \mathrm{Hom}_{\mathrm{Top}_*}(S^1, X)$$

topologized with the compact-open topology.

Both constructions are functorial, and so we obtain endofunctors

$$\Sigma : \mathrm{Top}_* \to \mathrm{Top}_*, \qquad \Omega : \mathrm{Top}_* \to \mathrm{Top}_*$$

that "descend" (see Remark below) to functors on the homotopy categories

$$\Sigma : \mathrm{HoTop}_* \to \mathrm{HoTop}_*, \qquad \Omega : \mathrm{HoTop}_* \to \mathrm{HoTop}_* .$$

Remark 2.14. One needs to be careful here; for example, Σ still works well on CW complexes but for general topological spaces it's a bit tricky to define what Σ does properly in HoTop_*. Viceversa, Ω behaves well but in general Ω of a CW complex might not have a nice CW structure, rather it is only homotopy equivalent to a CW complex, hence we stay in Top_* and HoTop_*.

Fact 2.15. *Σ is left adjoint to Ω, both in Top_* (CW_*) and in HoTop_* (HoCW_*). Though, they are never equivalences.*

For example, this implies that $\pi_n(\Omega X) \simeq \pi_{n+1}(X)$. (Prove this!)
And now finally we can state the *Freudenthal suspension theorem*.

Theorem 2.16 (Freudenthal)**.** *Let $n \geq 1$ X be a pointed topological space that is n-*connected*, i.e. such that $\pi_k(X) = 0$ for all $k \leq n$. Then, the map*

$$\pi_k(X) \to \pi_k(\Omega\Sigma X) \simeq \pi_{k+1}(\Sigma X)$$

induced by the unit $X \to \Omega\Sigma X$ is an isomorphism if $k \leq 2n$, and an epimorphism if $k = 2n + 1$.

The map described above is sending a (pointed homotopy class of a) map $S^k \to X$ to the (pointed homotopy class of the) suspended map $\Sigma S^k \simeq S^{k+1} \to \Sigma X$.
Now, why is this result nice? The point is that we have

Fact 2.17. *If X is n-connected, then ΣX is $n + 1$-connected.*

therefore fixing k in the Theorem, for sufficiently large n, we will have $k + n \leq 2n$. So, the Theorem eventually applies to $\Sigma^n X$, which will be n-connected if we start with an X that is 0-connected (i.e. path connected), implying that $\pi_{k+n}(\Sigma^n X) \cong \pi_{k+m}(\Sigma^m X)$ for all $m \geq n$. We can therefore define the following.

Definition 2.18. Let X be a pointed topological space. The *stable homotopy groups* of X are defined as

$$\pi_k^S(X) := \mathrm{colim}_{n\to\infty}\, \pi_{k+n}(\Sigma^n X)$$

for all $k \geq 0$.

Note that these are always abelian groups. By the discussion before, they eventually stabilize, so the colimit will be equal to this.

Example 2.19. The stable homotopy groups of spheres are $\pi_k^S(S^0)$. These are the abelian groups that the homotopy groups of spheres $pi_{k+n}(S^n)$ eventually stabilize to.

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The stable homotopy groups of some X are also very hard to compute, but sometimes a bit easier, and for some things much better behaved. For example, note how they are all abelian groups. This leads to a first motivation to introduce a category of "spectra":

Motivation 2.20 (Motivation for spectra coming from homotopy theory)**.** Goal: have a category Sp that is "similar" to Top, but better behaved, in order to work well with spaces "up to stabilization". Among other things, we would like to have the following:

- functors
$$\begin{tikzcd} \mathrm{Top} \arrow[r, bend left, "\Sigma^\infty_+"] & \mathrm{Sp} \arrow[l, bend left, "\Omega^\infty"] \end{tikzcd}$$
that are adjoint with each other (Σ^∞_+ left adjoint, Ω^∞ right adjoint);

- a notion of homotopy and homotopy groups in Sp so that, if $X \in$ Top, then $\pi_k(\Sigma^\infty_+ X) \cong \pi^S_k(X_+)$. That is, the natural notion of homotopy groups in Sp should recover the stable homotopy groups of a space X when applied to $\Sigma^\infty_+ X$;

- endofunctors Σ, Ω on Sp, that are compatible with respectively Σ, Ω endofunctors in Top, via respectively $\Sigma^\infty_+, \Omega^\infty$;

- Σ, Ω on Sp should moreover be equivalences. For example, from this it would follow that we can define also homotopy groups for negative integers, and all of these would be abelian groups.

Coming up with such a category was probably the first time spectra appeared, in the form of prespectra.

2.2 Motivation 2 from algebra

In this second section, we will look at the more algebraic motivation for spectra: (co)corepresenting generalized (co)homology theories. [6]

Classically, recall that other powerful invariants of a topological space are given by homology and cohomology theories: given a topological space X and an abelian group G, one can define, for every n, abelian groups $\tilde{H}_n(X;G)$ and $\tilde{H}^n(X;G)$, respectively homology and cohomology, that are invariant up to homotopy. In fact, these can be extended to functors

$$\tilde{H}_n : \mathrm{Top}_* \to \mathrm{Ab}, \qquad \tilde{H}^n : \mathrm{Top}_*^{\mathrm{op}} \to \mathrm{Ab}$$

for every $n \geq 0$, that factor through $\mathrm{Top}_* \to \mathrm{HoTop}_*$. One can even extend these to functors being evaluated on pairs of topological spaces, and equip them with some "connecting" morphisms: this leads to the characterization of such (co)homology theories by the Eilenberg-Steenrod axioms, and is discussed a bit later on in this section.

Now, homology and cohomology are also invariants of topological spaces, just like homotopy groups, but are quite different in their nature (though connected in many deep ways). Better for some things, worse for others.

- On one side, they were conceived to be "easy" to compute, at least compared to homotopy groups, and in fact the Eilenberg-Steenrod axioms highlight this tendency towards computability very clearly. It is fairly straightforward to compute homology and cohomology of many classical topological spaces, for example for the spheres S^n one has:
$$\tilde{H}_k(S^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & k = n \\ 0 & \text{else} \end{cases}$$
and similarly for integral cohomology, whereas as we saw in the previous section, $\pi_k(S^n)$ is very much unpredictable for large k.

- On the other side, as invariants they tend to be less powerful/rich compared to homotopy groups. For example, homotopy groups are a "strong invariant" on CW complexes because they can detect homotopy equivalences: see Whitehead's Theorem 2.9.

[6] note that this section is all about reduced theories and pointed spaces, else some things don't quite work at level 0; I haven't even mentioned what this means but it shouldn't be important

Classically, (co)homology theories are defined via building an appropriate (co)chain complex and then taking (co)homology in the homological algebra sense. For example, integral homology $\tilde{H}^n(-,\mathbb{Z})$, as a functor Top $\to$ Ab, is obtained via the following sequence of functors:

$$\text{Top} \xrightarrow{\text{Sing}(-)} \text{sSet} \xrightarrow{\mathbb{Z}[-]} \text{Ch}_{\geq 0} \xrightarrow{\tilde{H}_n(-)} \text{Ab}$$

So, first one builds a simplicial set (the construction here is precisely the one we saw in Talk 1, the Sing of a topological space). Then, one can obtain a chain complex of abelian groups by taking $\mathbb{Z}[-]$ at every level and then defining differentials as alternating sums of face maps. Finally, one can then take homology of this complex.

It should be noted that then one obtains such a functor for every $n \geq 0$, that this can be extended to functors taking values in pairs of topological spaces (cfr relative homology), and that there are specific connecting morphisms that link consecutive levels of homology groups. I'm again omitting details as we don't really have time to discuss them, but you can look up the key words.

It turns out that (co)homology theories[7] are completely characterized by the so-called Eilenberg-Steenrod axioms: each (co)homology theory for some G as above satisfies these, and viceversa any data as above satisfying these axioms gives a (co)homology theory for some G. They are the following:

Axioms 2.21 (Eilenberg-Steenrod). *(I'm not spelling these out as the only important one atm is the last one)*

1. *Homotopy invariance*
2. *Excision*
3. *Additivity*
4. *Exactness*
5. *Dimension axiom: $\tilde{H}_n(S^0)$ (resp. $\tilde{H}^n(S^0)$) is 0 for all $n \neq 0$.*

Remark 2.22. From the above axioms it follows in particular that there is a natural isomorphism $\tilde{H}^{n+1}(\Sigma X; G) \cong \tilde{H}^n(X; G)$, and similarly for homology. This is why we work with reduced theories (else this fails for $n = 0$) and pointed spaces. This is moreover particularly noteworthy as it means that homology and cohomology behave well with respect to suspension... (don't worry, this is just some more foreshadowing)

Now, $\tilde{H}_*(-; G)$ and $\tilde{H}^*(-; G)$ satisfy these, and viceversa, and everyone's happy. But historically at some point, people started coming up with "(co)homology-like" theories that satisfy all of the above, except for the Dimension axiom. This lead to defining *generalized (co)homology theories* (sometimes referred to as exceptional) as satisfying all of the above axioms except the Dimension axiom.

Example 2.23. The following are examples of generalized (co)homology theories:

1. topological K-theory;
2. (co)bordism theories;
3. stable homotopy groups.

Let's look at cohomology now. It turns out that the usual cohomology $\tilde{H}^n(-; G)$ is representable as a functor HoTop $\to$ Ab:

Fact 2.24. *For all abelian groups G, and for all $n \geq 0$, there exist (pointed) spaces $K(G, n)$ such that*

$$\tilde{H}^n(-; G) \cong [-, K(G, n)]_*.$$

These spaces $K(G, n)$ are called Eilenberg-MacLane spaces.

Let's collect some of the properties of these spaces.

Proposition 2.25. *The spaces $K(G, n)$, for all G and $n \geq 0$, satisfy the following:*

1. $\pi_k(K(G, n)) = \begin{cases} G & k = n \\ 0 & \textit{else} \end{cases}$

[7] as functors, one for every $n \geq 0$, from the category of pairs of topological spaces to Ab equipped with specific connecting morphisms

2. *there always exist spaces with the above property, and are unique up to (pointed) homotopy equivalence. We may also say that a space is a $K(G, n)$ if it satisfies the above;*
3. *$\Omega K(G, n+1)$ is a $K(G, n)$;*
4. *they can be constructed explicitly as CW complexes (or via CW approximation).*

Note that unicity up to homotopy equivalence justifies choosing a representative for every G and n and calling it $K(G, n)$, as was done above. Thus, fixing G and given a cohomology theory $\tilde{H}^*(-; G)$, we may choose representatives $K(G, n)$ for each n and collect them together: $\{K(G, n)\}_n$. Note that choices of homotopy equivalences $\Omega K(G, n+1) \simeq K(G, n)$, together with expressing cohomology as the represented functor, yield natural isomorphisms $\tilde{H}^{n+1}(\Sigma X; G) \cong \tilde{H}^n(X; G)$. (Exercise: prove this!) Remember this for later, as this data is an example of a choice of objects in a "nice" category of spectra!

So usual cohomology theories can be represented. But what about generalized cohomology theories?? Here's motivation 2 for spectra then!

Motivation 2.26. (Motivation for spectra coming from representing cohomology) Goal: have a category Sp that is "similar" to Ab, but such that:

- we have a functor $\mathrm{Ab} \xrightarrow{H} \mathrm{Sp}$ sending G to HG, the *Eilenberg-MacLane spectrum* of G;
- there is a correspondence between generalized cohomology theories and spectra, i.e. objects of Sp: every spectrum gives a generalized cohomology theory, and every generalized cohomology theory is represented (in some sense) by a spectrum[8];
- under this correspondence, HG corresponds to the usual cohomology theory $\tilde{H}^*(-; G)$.

A similar thing can be said for generalized homology theories, being corepresented by spectra.

Example 2.27. One would then get spectra (co)representing topological K-theory, (co)bordism, stable homotopy groups ...

2.3 Potential definitions of Sp and properties of HoSp

Here I discuss how one may define a category of spectra classically, and the main properties of the homotopy category of spectra.

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[8]usually, one first defines a suitable category of spectra, giving generalized cohomology theories in some natural sense as taking functors represented by "levels" of a spectrum, and then proves that every generalized cohomology theory arises in this way. These results are usually referred to as "Brown representability"